

On the Stability of Stochastic Parametrically Forced Equations with Rank One Forcing

Timothy Blass¹

L.A. Romero²

J.R. Torczynski³

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Abstract

We derive simplified formulas for analyzing the stability of stochastic parametrically forced linear systems. This extends the results in [2] where, assuming the stochastic excitation is small, the stability of such systems was computed using a weighted sum of the extended power spectral density over the eigenvalues of the unperturbed operator. In this paper, we show how to convert this to a sum over the residues of the extended power spectral density. For systems where the parametric forcing term is a rank one matrix, this leads to an enormous simplification.

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1 Introduction

In [2], we analyzed stochastically forced ODEs of the form

$$\mathbf{B}_0 \frac{d\mathbf{x}}{dt} = (\mathbf{A}_0 + \epsilon f(t) \mathbf{A}_1) \mathbf{x}, \quad (1.1)$$

where ϵ is a small parameter, $f(t)$ is a random function, and \mathbf{A}_0 and \mathbf{A}_1 are $N \times N$ matrices. In [2], it was assumed that \mathbf{B}_0 was the identity matrix \mathbf{I} , but in this paper we relax this assumption.

In that paper, we assumed that $f(t)$ could be written as $f(t) = \langle \mathbf{a}, \mathbf{s}(t) \rangle$ where \mathbf{a} is an n -dimensional vector, \mathbf{s} is the output of an n -dimensional vector Ornstein-Uhlenbeck process, and $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbb{C}^n . We analyzed the moment stability of this equation. That is, we determined under what conditions the various moments of $\mathbf{x}(t)$ remain bounded in time. Though our derivation assumed that $f(t)$ could be derived from a vector Ornstein-Uhlenbeck process, we showed that, to second order in ϵ , our stability criterion depended only on the extended power spectral density of $f(t)$. Here, the extended power spectral density $G(z)$ of $f(t)$ is defined as

¹Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
tblass@andrew.cmu.edu

²Computational Mathematics and Algorithms Department, Sandia National Laboratories, MS 1320, P.O. Box 5800, Albuquerque, NM 87123-1320, lromero@sandia.gov

³Fluid Sciences and Engineering Department, Sandia National Laboratories, MS 0840, P.O. Box 5800, Albuquerque, NM 87185-0840, jrtorcz@sandia.gov

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$$G(z) = \int_0^\infty R(\tau) e^{-z\tau} d\tau, \quad (1.2)$$

where $R(\tau)$ is the autocorrelation function of $f(t)$.

We now briefly review the results of [2]; a more thorough review is given in §2. We showed that, to second order in ϵ , the condition for the second moment to remain bounded is

$$\text{Re}[\lambda] = \text{Re}[\lambda_0 + \epsilon^2 \lambda_2] < 0, \quad (1.3)$$

where

$$\lambda_0 = \sigma_p + \sigma_q \quad (1.4)$$

and σ_k are the eigenvalues of \mathbf{A}_0 (if the unperturbed system is stable, then $\text{Re}[\sigma_k] < 0$). The eigenvalues σ_p and σ_q are chosen so as to maximize the real part of the sum. More precisely, we want to choose σ_p and σ_q so as to minimize the value of ϵ that makes the sum in Eqn. (1.3) positive. For given values of σ_p and σ_q , the expression for λ_2 involves terms of the form

$$I_p = \sum_{k=1}^N \chi_{pk} \chi_{kp} G(\sigma_p - \sigma_k), \quad (1.5)$$

where χ_{ij} are coefficients that depend on inner products of the j th adjoint eigenvector of \mathbf{A}_0 with \mathbf{A}_1 multiplied by the i th eigenvector of \mathbf{A}_0 .

In this paper, we show that, if \mathbf{A}_1 is a rank one matrix ($\mathbf{A}_1 = \mathbf{u}\mathbf{v}^T$), then we can convert the sum in Eqn. (1.5) to a sum over the residues of G . We show that, if \mathbf{A}_1 is rank one, then the characteristic equation for the eigenvalues of $\mathbf{A}_0 + \epsilon \mathbf{A}_1$ can always be written as

$$\det(\sigma \mathbf{B}_0 - \mathbf{A}_0 - \epsilon \mathbf{A}_1) = 0 \quad \text{iff} \quad g_A(\sigma, \epsilon) = 0, \quad (1.6)$$

where

$$g_A(\sigma, \epsilon) = f_A(\sigma) + \epsilon. \quad (1.7)$$

In §4, we give an explicit formula for $f_A(\sigma)$. For now, we merely note that in this case the sum I_p can be written as

$$I_p = \frac{1}{f'_A(\sigma_p)} \sum_m \frac{1}{f_A(\sigma_p - \mu_m)} r_m. \quad (1.8)$$

Here, μ_m are the poles of the extended power spectral density $G(z)$, and r_m are the residues of G at these poles.

The results in [2] were derived for ordinary differential equations. However, the stability criterion (up to second order) can be expressed in terms of quantities such as eigenvalues, eigenfunctions, and inner products that carry over to partial differential equations. Hence, it is not unreasonable to expect that the results carry over to partial differential equations. If we make this assumption, then, in order to apply the results in [2], we need to evaluate infinite sums. However, if our function G has a finite number of poles, the results in this paper allow us to convert the infinite sum into a finite sum over the residues at the poles of G .

The formulation in this paper was arrived at by considering the stability of stochastically forced Faraday waves (i.e., standing waves in the liquid-gas interface in a vertically oscillating container). For small values of the liquid viscosity, it is possible to approximate the equation for the height of the free surface using the Mathieu equation. However, for more viscous waves, it is necessary to solve a partial differential equation to determine the height. We show that the formulation given in this paper applies to this problem and enormously simplifies the calculations (cf. Remark 6.1). However, the purpose of this paper is to present the mathematical formalism

for doing this, not to investigate the parameter space in the Faraday wave problem. In a later paper, we will more fully discuss the problem of stochastic Faraday waves.

In §2, we briefly review the results from [2]. We present these results for general symmetric positive definite mass matrices \mathbf{B}_0 and justify this in Appendix A. In §3, we show how to convert the sum over the eigenvalues of \mathbf{A}_0 in Eqn. (1.5) to a sum over the poles of G for arbitrary (not only rank one) matrices \mathbf{A}_1 . This conversion is particularly simple when \mathbf{A}_1 is a rank one matrix, and we show how to do this in §4. In §5, we discuss a simple mechanical example where \mathbf{A}_1 is rank one. In §6, we show how these results apply to stochastically induced Faraday waves.

2 Summary of Previous Work

In [2], we analyzed the stability of Eqn. (1.1) assuming that \mathbf{B}_0 was the identity matrix. In this section we summarize those results. However, we present the results for the more general case where \mathbf{B}_0 is assumed to be a symmetric positive definite matrix. The justification for applying the results in [2] to more general mass matrices is straightforward and is given in Appendix A.

In analyzing the stability of Eqn. (1.1), we suppose that the pair $(\mathbf{A}_0, \mathbf{B}_0)$ has generalized eigenvalues σ_k , $k = 1, \dots, N$, generalized eigenvectors ϕ_k , $k = 1, \dots, N$ and adjoint eigenvectors ψ_k , $k = 1, \dots, N$:

$$\mathbf{A}_0 \phi_k = \sigma_k \mathbf{B}_0 \phi_k, \quad \mathbf{A}_0^* \psi_k = \bar{\sigma}_k \mathbf{B}_0 \psi_k, \quad k = 1, \dots, N, \quad (2.1)$$

where the eigenvectors are normalized with respect to the inner product $\langle \mathbf{v}, \mathbf{u} \rangle_B = \bar{\mathbf{v}}^T \mathbf{B}_0 \mathbf{u}$, so that

$$\langle \psi_i, \phi_j \rangle_B = \delta_{ij}. \quad (2.2)$$

In [2], we showed that, for given values of σ_p and σ_q , the parameter λ_2 in Eqn. (1.3) could be written as

$$\lambda_2 = 8 \sum_{j,k=1}^N \frac{C_{jkpq} C_{pqjk}}{1 + \delta_{pq}} G(\sigma_p + \sigma_q - \sigma_j - \sigma_k), \quad (2.3)$$

where

$$C_{jk\ell m} = \frac{1}{4} (\delta_{jm} \chi_{k\ell} + \delta_{km} \chi_{j\ell} + \delta_{j\ell} \chi_{km} + \delta_{k\ell} \chi_{jm}), \quad (2.4)$$

$$\chi_{ij} = \langle \psi_i, \mathbf{A}_1 \phi_j \rangle, \quad (2.5)$$

and $\langle \cdot, \cdot \rangle$ is the standard inner product $\langle \mathbf{v}, \mathbf{u} \rangle = \bar{\mathbf{v}}^T \mathbf{u}$.

Though Eqn. (2.3) was derived under the assumption that $f(t)$ was the output from a vector Ornstein-Uhlenbeck process, the fact that our answer depends only on the extended power spectral density of the process strongly suggests that we should be able to apply this formula more generally. We make this assumption in this paper and do not limit our analysis to such processes.

Though Eqn. (2.4) is a compact way of writing the equation for λ_2 , we prefer to expand the Kronecker delta functions in this equation to get the alternative form

$$\lambda_2 = \frac{1}{2(1 + \delta_{pq})} \left(4\chi_{pp}\chi_{qq}G(0) + 2\chi_{pq}\chi_{qp}(G(\sigma_p - \sigma_q) + G(\sigma_q - \sigma_p)) + 2(I_p + I_q + 2\delta_{pq}I_p) \right), \quad (2.6)$$

where I_p is given as in Eqn. (1.5).

The main simplification in this paper comes from finding an alternative expression for I_p and an expression for the products of the χ_{ij} that does not require computing eigenvectors, adjoint eigenvectors, and inner products.

3 Converting the Sum I_p

To compute λ_2 , we must compute I_p in Eqn. (1.5), where the sum is taken over all of the generalized eigenvalues of $(\mathbf{A}_0, \mathbf{B}_0)$. For large systems, this sum can be cumbersome to compute because one needs to compute all of the eigenvalues, eigenvectors, adjoint eigenvectors, and coefficients $\chi_{pk}\chi_{kp}$. In the case of PDEs, the computation of I_p is further complicated by the fact that there are infinitely many eigenvalues, etc., so one needs to decide how to truncate the series without losing accuracy. We derive an alternative expression for I_p in two stages. First, we show that the coefficients $\chi_{pk}\chi_{kp}$ can be computed in terms of an auxiliary function that determines an *equivalent characteristic equation*, which we now define.

Definition 3.1. Given a pair of matrices (\mathbf{K}, \mathbf{M}) , we say that a function $g(\sigma)$ determines an equivalent characteristic equation for (\mathbf{K}, \mathbf{M}) provided $g(\sigma) = 0$ if and only if $\det(\sigma\mathbf{M} - \mathbf{K}) = 0$. We refer to such functions g as equivalent characteristic functions.

An equivalent characteristic equation allows us to replace $\chi_{pk}\chi_{kp}$ in Eqn. (1.5) with a more convenient expression (cf. Eqn (3.18)). The second stage is to then convert the expression for I_p from a sum over the generalized eigenvalues of $(\mathbf{A}_0, \mathbf{B}_0)$ into a sum over the poles of G . This is done assuming G is meromorphic and $f'_p(\sigma_k) \neq 0$ for each k , so that an argument involving contour integration can be applied.

3.1 Alternative Expression for $\chi_{pk}\chi_{kp}$

Using the standard inner product, we can write

$$\chi_{pk}\chi_{kp} = \overline{\psi}_k^T \mathbf{A}_1 \phi_p \overline{\psi}_p^T \mathbf{A}_1 \phi_k = \langle \psi_k, \mathbf{L}_p \phi_k \rangle, \quad (3.1)$$

where

$$\mathbf{L}_p = \mathbf{a}_p \mathbf{b}_p^T \quad (3.2)$$

and

$$\mathbf{a}_p = \mathbf{A}_1 \phi_p, \quad \mathbf{b}_p^T = \overline{\psi}_p^T \mathbf{A}_1. \quad (3.3)$$

We define

$$\mathbf{A}_p(\epsilon) = \mathbf{A}_0 + \epsilon \mathbf{L}_p = \mathbf{A}_0 + \epsilon \mathbf{a}_p \mathbf{b}_p^T \quad (3.4)$$

and denote the generalized eigenvalues and eigenvectors of $(\mathbf{A}_p, \mathbf{B}_0)$ by $\hat{\sigma}_k$ and $\hat{\phi}_k$. That is, $\hat{\sigma}_k(\epsilon)$ is the eigenvalue satisfying

$$\mathbf{A}_p(\epsilon) \hat{\phi}_k = \hat{\sigma}_k \mathbf{B}_0 \hat{\phi}_k. \quad (3.5)$$

Noting that, at $\epsilon = 0$, $\hat{\psi}_p(0) = \psi_p$, $\hat{\phi}_p(0) = \phi_p$, we can differentiate Eqn. (3.5) with respect to ϵ , evaluate at $\epsilon = 0$, left-multiply by $\overline{\psi}_p^T$ and use $\overline{\psi}_p^T (\sigma_p \mathbf{B}_0 - \mathbf{A}_0) = 0$ to obtain the well known result from the perturbation theory of eigenvalues

$$\left. \frac{d\hat{\sigma}_k}{d\epsilon} \right|_{\epsilon=0} = \frac{\overline{\psi}_p^T \mathbf{L}_p \phi_p}{\overline{\psi}_p^T \mathbf{B}_0 \phi_p}. \quad (3.6)$$

Combining Eqn. (3.6) with Eqn. (3.1) and the normalization in Eqn. (2.2), we have

$$\chi_{pk}\chi_{kp} = \left. \frac{d\hat{\sigma}_k}{d\epsilon} \right|_{\epsilon=0}, \quad (3.7)$$

so we can write the sum in Eqn. (1.5) as

$$I_p = \sum_{k=1}^N \left(\left. \frac{d\hat{\sigma}_k}{d\epsilon} \right|_{\epsilon=0} \right) G(\sigma_p - \sigma_k). \quad (3.8)$$

If the eigenvalues $\hat{\sigma}(\epsilon)$ satisfy the equivalent characteristic equation

$$g_p(\hat{\sigma}, \epsilon) = 0 \quad (3.9)$$

for some equivalent characteristic function g_p , then implicitly differentiating Eqn. (3.9), evaluating the result at $\epsilon = 0$, and solving for $d\hat{\sigma}/d\epsilon$ gives

$$\left. \frac{d\hat{\sigma}_k}{d\epsilon} \right|_{\epsilon=0} = -\frac{\frac{\partial g_p}{\partial \epsilon}(\sigma_k, 0)}{\frac{\partial g_p}{\partial \sigma}(\sigma_k, 0)}. \quad (3.10)$$

On the right side of Eqn. (3.10), we are using $\hat{\sigma}_k(0) = \sigma_k$, where σ_k are the generalized eigenvalues of $(\mathbf{A}_0, \mathbf{B}_0)$.

The following lemma shows that, for matrices of the form $\mathbf{A}_p(\epsilon)$ as in Eqn. (3.4), there is always an equivalent characteristic equation with a particularly simple form.

Lemma 3.1. *The eigenvalues $\hat{\sigma}(\epsilon)$ of $\mathbf{A}_p(\epsilon)$ in Eqn. (3.4) satisfy the equivalent characteristic equation*

$$g_p(\hat{\sigma}, \epsilon) := f_p(\hat{\sigma}) + \epsilon = 0, \quad (3.11)$$

where

$$\frac{1}{f_p(\hat{\sigma})} = -\mathbf{b}_p^T (\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0)^{-1} \mathbf{a}_p. \quad (3.12)$$

Remark 3.1. For any $\hat{\sigma}$ that is not a generalized eigenvalue of $(\mathbf{A}_0, \mathbf{B}_0)$, the formula in Eqn. (3.12) is well-defined. In fact, f_p is well-defined even at each generalized eigenvalue σ_k where $f_p(\sigma_k) = 0$. Denoting the adjugate matrix (the transpose of the cofactor matrix) of $\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0$ by $\text{adj}(\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0)$, Cramer's rule gives $\text{adj}(\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0) = \det(\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0)(\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0)^{-1}$ whenever $\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0$ is non-singular. If we take $f_p(\hat{\sigma}) = -\det(\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0)/(\mathbf{b}_p^T \text{adj}(\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0) \mathbf{a}_p)$ as the definition of f_p , then it agrees with Eqn. (3.12) for any $\hat{\sigma} \neq \sigma_k$, and shows that $f_p(\sigma_k) = 0$.

Proof. The eigenvalues $\hat{\sigma}$ satisfy

$$h_p(\hat{\sigma}, \epsilon) = \det(\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0 - \epsilon \mathbf{a}_p \mathbf{b}_p^T) = 0. \quad (3.13)$$

If we define

$$h_0(\hat{\sigma}) := \det(\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0), \quad (3.14)$$

then using the formula for the determinant of the rank one update of the identity matrix,

$$\det(\mathbf{I} + \mathbf{u} \mathbf{v}^T) = 1 + \mathbf{v}^T \mathbf{u}, \quad (3.15)$$

we see that the eigenvalues satisfy

$$0 = h_p(\hat{\sigma}, \epsilon) = h_0(\hat{\sigma}) \left(1 - \epsilon \mathbf{b}_p^T (\hat{\sigma} \mathbf{B}_0 - \mathbf{A}_0)^{-1} \mathbf{a}_p \right) = h_0(\hat{\sigma}) + \epsilon \frac{h_0(\hat{\sigma})}{f_p(\hat{\sigma})}, \quad (3.16)$$

where $f_p(\hat{\sigma})$ is defined as in Eqn. (3.12). Note that using Cramer's rule makes it simple to see that $h_0(\hat{\sigma})/f_p(\hat{\sigma})$ is never singular, so, if we divide $h_p(\hat{\sigma})$ by this quantity, we do not introduce any new zeros. Dividing Eqn. (3.16) by $h_0(\hat{\sigma})/f_p(\hat{\sigma})$ yields the equation for $\hat{\sigma}$ as in Eqn. (3.11). \square

Implicitly differentiating equation (3.11) with respect to ϵ and evaluating the result at $\epsilon = 0$ and $\sigma = \sigma_k$ we find that

$$\left. \frac{d\hat{\sigma}_k}{d\epsilon} \right|_{\epsilon=0} = -\frac{1}{f'_p(\sigma_k)}. \quad (3.17)$$

Combining this with Eqn. (3.7) gives the following lemma.

Lemma 3.2. *With χ_{ij} defined as in Eqn. (2.5), and f_p as in Eqn. (3.12), we have*

$$\chi_{pk}\chi_{kp} = -\frac{1}{f'_p(\sigma_k)}. \quad (3.18)$$

The fact that g_p depends linearly on ϵ is due to the fact that \mathbf{A}_p is a rank one update of \mathbf{A}_0 . The choice of \mathbf{A}_p is motivated by the form of $\chi_{pk}\chi_{kp}$ in Eqn. (3.1) and is related to the original problem Eqn. (1.1) only through the stability analysis (and specifically, the form of I_p). No assumptions have been made on \mathbf{A}_1 at this point, so the formulas for g_p and f_p hold in general. The only obstacle to using Eqn. (3.18) to compute $\chi_{pk}\chi_{kp}$ is that f_p (and f'_p) may be difficult to compute. In particular, we need to compute the eigenvector and the adjoint eigenvector of σ_p and take some inner products. In §4, we show that if \mathbf{A}_1 is rank one, then the same algebraic manipulations can be performed to find an equivalent characteristic function g_A that is simple to compute (i.e., it does not involve the generalized eigenfunctions and adjoint eigenfunctions) and leads to a simple formula for f_p .

3.2 Rewriting I_p as a Sum over the Poles of G

Using Lem. 3.2 in Eqn. (1.5), we can write

$$I_p = -\sum_{k=1}^N \frac{1}{f'_p(\sigma_k)} G(\sigma_p - \sigma_k). \quad (3.19)$$

Cramer's rule can be used to show that $1/f_p(\sigma)$ decays at least as fast as $1/\sigma$ as $|\sigma|$ approaches ∞ . The form of the function $G(\sigma)$ in Eqn. (9.2) shows that it also decays at least as fast as $1/\sigma$ for large values of $|\sigma|$. With this in mind,

$$0 = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{C_R} \frac{G(\sigma_p - \sigma)}{f_p(\sigma)} d\sigma, \quad (3.20)$$

where $C_R := \{\sigma \in \mathbb{C} : |\sigma| = R\}$. We denote the poles of $G(z)$ by μ_m , $m = 1, \dots, M$ and the residue of G at μ_m by r_m , for each m . Thus, $-r_m$ is the residue of $G(\sigma_p - \sigma)$ at $\sigma_p - \mu_m$. Now, applying the Residue theorem to (3.20), and recalling that the zeros of f_p are σ_k , we see that

$$0 = \sum_{k=1}^N \frac{G(\sigma_p - \sigma_k)}{f'_p(\sigma_k)} - \sum_{m=1}^M \frac{r_m}{f_p(\sigma_p - \mu_m)} \quad (3.21)$$

assuming $f'_p(\sigma_k) \neq 0$ for each k . This leads to the following theorem.

Theorem 3.3. *Let $f_p(\sigma)$ be defined as in Eqn. (3.12). If G is meromorphic and $f'_p(\sigma_k) \neq 0$ for each k , then the sum I_p in Eqn. (1.5) can be written as*

$$I_p = -\sum_m \frac{r_m}{f_p(\sigma_p - \mu_m)}. \quad (3.22)$$

Here, the sum is taken over all of the poles μ_m of $G(z)$, and r_m is the residue of $G(z)$ at the pole μ_m .

Thm. 3.3 provides a simple expression for I_p provided there is a practical way to compute the function f_p . We show in the next section that when \mathbf{A}_1 is rank one, then it is simple to compute f_p , and hence it is simple to compute λ_2 and determine the moment stability of the system.

Remark 3.2. We note that Thm. 3.3 applies to any system of the form (1.1), including systems that are a priori unstable (i.e., if $f_p(\sigma) = 0$ has solutions in the right half-plane). However, this scenario is perhaps less interesting because in order to stabilize a system with stochastic forcing, one would expect to use a large value of ϵ , requiring a different approach than presented here. We are interested in applying Thm. 3.3 to systems that are stable for $\epsilon = 0$ (i.e., $\text{Re}[\sigma_k] < 0$ for all k), specifically, the Faraday wave problem discussed in §6.

4 Case where \mathbf{A}_1 is Rank One

We now suppose that \mathbf{A}_1 is rank one and hence has the form

$$\mathbf{A}_1 = \mathbf{u}\mathbf{v}^T. \quad (4.1)$$

Thus, $\mathbf{A}_0 + \epsilon\mathbf{A}_1$ is a rank one update of \mathbf{A}_0 , just like \mathbf{A}_p was in Eqn. (3.4). The same algebraic manipulations that were applied to $\det(\sigma\mathbf{B}_0 - \mathbf{A}_p)$ in the proof of Lem. 3.1 yields an equivalent characteristic equation for $\det(\sigma\mathbf{B}_0 - \mathbf{A}_0 - \epsilon\mathbf{A}_1)$ given by $g_A(\sigma, \epsilon)$, where $g_A(\sigma)$ is defined as in Eqn. (1.7) and

$$\frac{1}{f_A(\sigma)} = -\mathbf{v}^T (\sigma\mathbf{B}_0 - \mathbf{A}_0)^{-1} \mathbf{u}. \quad (4.2)$$

The following lemma shows that we can express the function $f_p(\sigma)$ in Eqn. (3.12) in terms of $f_A(\sigma)$.

Lemma 4.1. *Assuming $\mathbf{A}_1 = \mathbf{u}\mathbf{v}^T$, then the function $f_p(\sigma)$ in Eqn. (3.12) can be written as*

$$f_p(\sigma) = -f_A(\sigma)f'_A(\sigma_p). \quad (4.3)$$

Proof. Using Eqns. (3.12) and (3.3) and our expression in Eqn. (4.1) for \mathbf{A}_1 , a simple calculation shows that

$$f_p(\sigma) = \frac{1}{\left(\overline{\psi}_p^T \mathbf{u}\mathbf{v}^T \phi_p\right)} f_A(\sigma). \quad (4.4)$$

To prove the theorem, it is only necessary to show that

$$\overline{\psi}_p^T \mathbf{u}\mathbf{v}^T \phi_p = -\frac{1}{f'_A(\sigma_p)}. \quad (4.5)$$

Since $(\mathbf{A}_0 + \epsilon\mathbf{A}_1)\phi_p(\epsilon) = \sigma_p(\epsilon)\mathbf{B}_0\phi_p(\epsilon)$, if we differentiate and left-multiply by $\overline{\psi}_p^T$, we see that

$$\frac{d}{d\epsilon}\sigma_p|_{\epsilon=0} = \overline{\psi}_p^T \mathbf{A}_1 \phi_p = \overline{\psi}_p^T \mathbf{u}\mathbf{v}^T \phi_p \quad (4.6)$$

(analogous to Eqn. (3.6)). Since $\sigma_p(\epsilon)$ satisfies Eqn. (1.7), the formula in Eqn. (4.5) now follows by implicitly differentiating Eqn. (1.7). \square

Combining Thm. 3.3 with Lem. 4.1 yields the following theorem.

Theorem 4.2. *If \mathbf{A}_1 is a rank one matrix (and hence $\mathbf{A}_1 = \mathbf{u}\mathbf{v}^T$), then the roots of $\det(\sigma\mathbf{B}_0 - \mathbf{A}_0 - \epsilon\mathbf{A}_1)$ are the same as the roots of $g_A(\sigma) = f_A(\sigma) + \epsilon$, where $f_A(\sigma)$ is defined as in Eqn. (4.2). Furthermore, under the assumptions of Thm. 3.3, the sum I_p in Eqn. (1.5) can be written as in Eqn. (1.8). Here, the sum is taken over all of the poles μ_m of $G(z)$ and r_m is the residue of $G(z)$ at the pole μ_m .*

In order to evaluate λ_2 using Eqn. (2.6), it is necessary to evaluate $\chi_{pq}\chi_{qp}$, and $\chi_{pp}\chi_{qq}$. Using Lems. 3.2 and 4.1, we see that

$$\chi_{pq}\chi_{qp} = \frac{1}{f'_A(\sigma_q)f'_A(\sigma_p)}. \quad (4.7)$$

Using Eqn. (4.5) and Eqn. (2.5), we see that

$$\chi_{pp} = -\frac{1}{f'_A(\sigma_p)}. \quad (4.8)$$

Eqns. (4.7) and (4.8) yield the following theorem.

Theorem 4.3. *Assuming \mathbf{A}_1 is a rank one matrix and that $f_A(\sigma)$ is defined as in Eqn. (4.2) and satisfies $f'_A(\sigma_k) \neq 0$ for $k = p, q$, we have*

$$\chi_{pp}\chi_{qq} = \chi_{pq}\chi_{qp} = \frac{1}{f'_A(\sigma_q)f'_A(\sigma_p)}. \quad (4.9)$$

Thus, if \mathbf{A}_1 is rank one, the terms in the expression for λ_2 given in Eqn. (2.6) can all be computed in terms of f_A , σ_p , σ_q , and the poles and residues of G . Moreover, $f_A(\sigma)$ is straightforward to compute, as it only requires knowing \mathbf{B}_0 , \mathbf{A}_0 , and \mathbf{A}_1 .

5 A Simple Mechanical System

In this section, we consider a simple mechanical example where \mathbf{A}_1 is rank one and hence allows us to use Thms. 4.2 and 4.3.

We consider a pendulum attached to a support (the pivot) constrained to move along a horizontal line (x direction) that is vibrated up and down vertically (z direction). The motion of this system is described in the accelerated reference frame in which this line is fixed. In this frame, the system experiences an effective time-varying gravitational field given by the line's vertical acceleration. We denote the x coordinate of the support by ξ and the angle of the pendulum relative to the downward (negative) z direction by θ . The support has a mass of m_S , the bob at the end of the pendulum has a mass of m_P , and the pendulum shaft is massless and has a length of ℓ . At any instant, the position of the support is denoted by \mathbf{r}_S , and the position of the pendulum bob is denoted by \mathbf{r}_P . These vectors are thus expressed as

$$\mathbf{r}_S = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \quad \mathbf{r}_P = \begin{pmatrix} \xi \\ 0 \end{pmatrix} - \ell \mathbf{e}_r(\theta), \quad \mathbf{e}_r = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}. \quad (5.1)$$

We suppose that the support is attached to a linear spring with spring constant K_S and that the mass on the pendulum is acted on by a spatially uniform gravitational field with acceleration constant g_0 . In the reference frame of the pendulum, when a vertical forcing is applied, the gravitational acceleration varies in time, which we write as $g(t) = g_0 + \epsilon f(t)$. The Lagrangian of this system can be written as

$$\mathcal{L} = \frac{1}{2}m_S\dot{\xi}^2 + \frac{1}{2}m_P \left| \dot{\xi} \mathbf{e}_x - \ell \dot{\theta} \frac{\partial \mathbf{e}_r}{\partial \theta} \right|^2 + m_P \ell g(t) \mathbf{e}_z \cdot \mathbf{e}_r(\theta) - \frac{1}{2}K_S \xi^2. \quad (5.2)$$

Here, \mathbf{e}_x and \mathbf{e}_z are the unit vectors in the x and z directions. We also assume that the spring attached to the support has damping γ_S and that the pendulum has damping γ_P . The linearized equations of motion about the equilibrium $\xi = 0$, $\theta = 0$ are given by

$$\mathbf{M}\ddot{\mathbf{z}} + \mathbf{C}\dot{\mathbf{z}} + \mathbf{K}\mathbf{z} = 0, \quad \mathbf{z} = \begin{pmatrix} \xi \\ \theta \end{pmatrix}, \quad (5.3)$$

where

$$\mathbf{M} = \begin{pmatrix} m_S + m_P & \ell m_P \\ \ell m_P & \ell^2 m_P \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \gamma_S & 0 \\ 0 & \gamma_P \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} K_S & 0 \\ 0 & g(t)\ell \end{pmatrix}. \quad (5.4)$$

Using $g(t) = g_0 + \epsilon f(t)$, we write $\mathbf{K} = \mathbf{K}_0 + \epsilon f(t)\mathbf{K}_1$, where $\mathbf{K}_0 = \text{diag}(K_S, g_0\ell)$ and $\mathbf{K}_1 = \text{diag}(0, \ell)$ are the unperturbed and perturbed parts of \mathbf{K} . Writing Eqn. (5.3) as a first-order system of equations, we obtain a system as in Eqn. (1.1) with

$$\mathbf{B}_0 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix}, \quad \mathbf{A}_0 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K}_0 & -\mathbf{C} \end{pmatrix}, \quad \mathbf{A}_1 = \mathbf{u}\mathbf{v}^T, \quad (5.5)$$

where $\mathbf{u} = (0, 0, 0, -\ell)^T$ and $\mathbf{v} = (0, 1, 0, 0)^T$.

Since \mathbf{A}_1 is rank one, we can apply the results of §4. Thus, we can compute f_p by computing f_A as in Eqn. (4.2). Indeed, the equivalent characteristic equation

$$h(\sigma) = \det(\sigma\mathbf{B}_0 - \mathbf{A}_0 - \epsilon\mathbf{A}_1) = \det(\sigma^2\mathbf{M} + \sigma\mathbf{C} + \mathbf{K}_0 + \epsilon\mathbf{K}_1) = 0 \quad (5.6)$$

depends linearly on ϵ and leads to an explicit formula for f_A . In particular, we have

$$h(\sigma) = h_0(\sigma) + \epsilon h_1(\sigma) \quad (5.7)$$

where

$$h_0(\sigma) := \sigma^4 \ell^2 m_P m_S + \sigma^3 (\gamma_P (m_P + m_S) + \ell^2 \gamma_S m_P) + \sigma^2 (\gamma_S \gamma_P + K_S \ell^2 m_P) + \sigma \gamma_P K_S + g_0 h_1(\sigma), \quad (5.8)$$

and

$$h_1(\sigma) := \sigma^2 \ell (m_P + m_S) + \sigma \gamma_S \ell + K_S \ell. \quad (5.9)$$

Finally, we arrive at an equivalent characteristic equation

$$h_A(\sigma) = f_A(\sigma) + \epsilon = 0, \quad \text{where} \quad f_A(\sigma) = \frac{h_0(\sigma)}{h_1(\sigma)}. \quad (5.10)$$

We have computed I_p for this system using Eqn. (1.5), which involves computing all of the eigenvalues, eigenvectors, and coefficients χ_{pk} using inner products. We have also computed I_p using Eqn. (1.8) which involves computing the poles and residues of G , $f_A(\sigma_p - \mu_m)$, and $f'_A(\sigma_p)$. These calculations were carried out using the extended power spectral density described in Appendix B. The two ways of doing the calculations agree to within machine precision. Having an explicit formula for f_A , as above, allows one to avoid any computation involving the eigenfunctions and adjoint eigenfunctions. The expression as a sum over the poles of G allows one to avoid computing sums with many terms, which is the case when Eqn. (1.1) is a large system. In this low-dimensional example, little effort is saved by using Eqn. (1.8), but, for larger systems and for PDEs, the formula for I_p in Eqn. (1.8) is much simpler to use than the formula in Eqn. (1.5).

6 Application to Stochastic Faraday Waves

Here, we briefly outline how Thms. 4.2 and 4.3 apply to the case of viscous capillary gravity waves and hence can be used to analyze stochastically induced Faraday waves.

For small amounts of damping, it is possible to model Faraday waves using the damped Mathieu equation. In [9, 7, 1], the stability of stochastically forced Faraday waves was analyzed using the Mathieu equation and the results in [8] for the stochastically forced Mathieu equation.

In [5, 3, 4, 6], deterministic viscous Faraday waves were analyzed without making the small-damping approximation. We analyze the stability of stochastically forced Faraday waves without making the small-damping approximation but instead using the full linearized Navier-Stokes equations.

Thus, we consider the linearized equations for viscous capillary gravity waves. We assume that our system is in a time-varying gravitational field. The time variation arises from the fact that the container holding the liquid is being moved up and down vertically and that the liquid motion is considered in a non-inertial frame of reference moving with the container.

We Fourier-transform these equations in the horizontal direction, and, without loss of generality, we assume that our disturbance is two-dimensional, depending only on the horizontal coordinate x and the vertical coordinate z . We assume that all quantities vary like

$$q(x, z, t) = q(z, t)e^{i\alpha x}. \quad (6.1)$$

With all of this in mind, the x and z components of the horizontally Fourier-transformed, linearized Navier-Stokes equations with the hydrostatic pressure subtracted are given by

$$\rho \frac{\partial u}{\partial t} + i\alpha p = \mu \left(\frac{d^2 u}{dz^2} - \alpha^2 u \right), \quad (6.2)$$

$$\rho \frac{\partial w}{\partial t} + \frac{dp}{dz} = \mu \left(\frac{d^2 w}{dz^2} - \alpha^2 w \right), \quad (6.3)$$

and the continuity equation is given by

$$i\alpha u + \frac{dw}{dz} = 0. \quad (6.4)$$

Here (u, v, p) are the horizontal velocity, the vertical velocity, and the pressure with the hydrostatic component removed.

We apply a no-slip boundary condition at $z = -L$:

$$u(-L, t) = w(-L, t) = 0. \quad (6.5)$$

If $h(t)$ is the spatially Fourier-transformed height of the free surface, the linearized kinematic boundary condition is given by

$$\frac{dh}{dt} = w \quad \text{at } z = 0. \quad (6.6)$$

The dynamic boundary conditions are given by

$$2\mu \frac{dw}{dz} - p = -(T\alpha^2 + \rho(g_0 + \epsilon f(t))) h \quad \text{at } z = 0, \quad (6.7)$$

$$\frac{du}{dz} + i\alpha w = 0 \quad \text{at } z = 0. \quad (6.8)$$

The function $f(t)$ in Eqn. (6.7) corresponds to the time-varying effective gravitational field caused by the vertical vibration of the container.

If we were to spatially discretize this system in the z direction, we would end up with a system of equations of the form

$$\mathbf{M} \frac{d\mathbf{q}}{dt} = \mathbf{K}_0 \mathbf{q} + \epsilon f(t) \mathbf{K}_1 \mathbf{q}, \quad (6.9)$$

where \mathbf{q} is a vector containing the discretization of (u, w, p) , along with h , and \mathbf{M} , \mathbf{K}_0 , and \mathbf{K}_1 are matrices that depend on the wavenumber α . Due to the continuity equation, the matrix

\mathbf{M} would be singular, thus yielding a system of differential algebraic equations with KKT-like structure (as discussed in Appendix C. In Appendix C, we show that the results in §2 apply to such systems.

It is crucial to note that the dependence on the parameter ϵ is rank one. That is, all occurrences of the parameter ϵ in our governing equations multiply the unknown h . In our discretized equations of motion, this situation would yield the term $\mathbf{K}_1 = \mathbf{a}\mathbf{b}^T$, where \mathbf{b} is a vector that is all zeros except for the component involving h . Thus, \mathbf{K}_1 is rank one.

When the depth L of the container is infinite, the dispersion relation for these equations is given by

$$g_D(\sigma, \epsilon) = f_D(\sigma) + \epsilon, \quad (6.10)$$

where

$$f_D(\sigma) = \frac{1}{\alpha} (\sigma + 2\nu\alpha^2)^2 - 4\alpha^2 (\nu)^{3/2} \sqrt{\sigma + \nu\alpha^2} + \frac{T}{\rho} \alpha^2 + g_0. \quad (6.11)$$

Since the dependence on g appears as a rank one term in the governing equations, this dispersion relation has the simple form that we expect to observe for a rank one system.

For finite values of L , the dispersion relation is more complicated, but it still has the simple form required of a rank one system. We give this dispersion relation in Eqn. (11.9) in Appendix D.

We have carried out calculations for stochastic Faraday waves using both Thm. 4.2 and the formulation in [2]. We use the following parameter values: a liquid density of $\rho = 0.95 \text{ g/cm}^3$, a liquid kinematic viscosity of $\nu = 0.1 \text{ cm}^2/\text{s}$, a liquid-gas surface tension of $T = 70 \text{ g/s}^2$, a steady gravitational acceleration of $g_0 = 1000 \text{ cm/s}^2$, and a wavenumber of $\alpha = 5 \text{ cm}^{-1}$. We consider the power spectral density described in Appendix B.

Both of these formulations give identical results. To illustrate this, Table 1 shows how values from using Eqn. (1.5) to evaluate I_p compare to values from using Thm. 4.2 to evaluate this sum. In the table, we calculate the sum in Eqn. (1.5) for different values of N . Clearly as N approaches infinity, the values from Eqn. (1.5) converge to the value from Thm. 4.2. It is clear that, as L becomes large, the number N of terms in Eqn. (1.5) must be increased to maintain the same accuracy.

Table 2 shows the difference between values from applying Thm. 4.2 to the finite-depth dispersion relation in Eqn. (11.9) for a given depth L relative to values from applying Thm. 4.2 to the infinite-depth dispersion relation in Eqn. (6.11). The finite-depth values clearly converge to the infinite-depth values as L approaches ∞ .

Remark 6.1. It is good to compare the cost of performing a brute-force calculation using Eqn. (1.5) versus applying Thm. 4.2. When making this comparison, one should take into account the effort of the analyst, not just the computational cost. Note that, for our system to be stable, it must be stable for all values of α . Hence, to do a stability analysis, it is necessary to sweep through all values of α to find the smallest value of ϵ needed to make some wavenumber unstable. For this reason, it is highly desirable to have a method that requires little intervention by the analyst. When computing the sum for I_p by brute force, we need to know all of the eigenvalues, eigenfunctions, and adjoint eigenfunctions, as well as how to compute the inner product to determine χ_{ij} . Although the eigenvalues can in principle be determined using the dispersion relation, one needs to have a good initial guess for the eigenvalues in order to apply Newton's method. Though this is not, in principle, a difficult thing to do, it greatly complicates the sweep through the wavenumber α . A considerable amount of theoretical effort is needed to compute the quantities χ_{ij} based on Eqn. (2.5).

On the other hand, in order to use Thm. 4.2, all that one needs is the dispersion relation for σ and the poles and residues of G . In particular, there is no need to calculate the eigenvalues, the eigenfunctions, or the inner products χ_{ij} . It should be noted that the ease of this method depends on the particular application. If one is interested in doing laboratory experiments on

N	error $L = 1$ cm	error $L = 2$ cm	error $L = 5$ cm	error $L = 10$ cm
5	6.6522e-03	7.9024e-03	8.1624e-03	8.1793e-03
10	2.1339e-03	6.1278e-03	7.9499e-03	8.1492e-03
20	6.2132e-05	1.8267e-03	6.7127e-03	7.9037e-03
40	4.9696e-07	5.3631e-05	3.0219e-03	6.6182e-03
80	3.9957e-09	4.5087e-07	1.9059e-04	2.9354e-03
160	4.2508e-11	3.3930e-09	2.0284e-06	1.8370e-04
320	7.1332e-13	2.5917e-11	1.5824e-08	1.9844e-06
640	1.9300e-14	1.9743e-13	1.2227e-10	1.5650e-08
1280	1.0307e-14	6.6570e-15	9.5273e-13	1.2160e-10

Table 1: This table gives the relative error in the quantity I_p calculated using Eqn. (1.5) with N eigenvalues compared to the exact expression calculated using Eqn. (1.8), where the sum is evaluated by taking the residues at the poles of G . The errors are given for different values of the depth L , where the physical parameters are $\rho = 0.95$ g/cm³, $\nu = 0.1$ cm²/s, $T = 70$ g/s², $g_0 = 1000$ cm/s², and $\alpha = 5$ cm⁻¹ and the power spectral density described in Appendix B is used.

	$L = 0.25$ cm	$L = 0.5$ cm	$L = 1$ cm	$L = 2$ cm	$L = 4$ cm
error	1.3459e-01	1.4134e-02	9.7424e-05	4.4170e-09	2.2693e-16

Table 2: This table gives the relative error in the quantity I_p calculated using the finite-depth dispersion relation in Eqn. (11.9) from Appendix D compared to the value calculated using the infinite-depth relation in Eqn. (6.11) for various depths L .

stochastic Faraday waves, then one has control over the power spectral density that is used. In this case, one could use the power spectral density described in Appendix D. However, if one is interested in applying the results to a power spectral density that has been measured experimentally, then it is necessary to do some sort of rational function approximation to $G(z)$ first and to compute its poles subsequently. In this case, applying Thm. 4.2 would require some preliminary steps.

7 Conclusions

We have derived formulas that allow us to evaluate the stability of parametrically forced stochastic equations. When the stochastic forcing is multiplying a rank one matrix, this leads to an enormous simplification over a brute-force implementation of the expressions in [2], especially when the dimension of the system is large. We have shown how our results apply to a simple mechanical example, as well as to stochastically forced Faraday waves.

8 Appendix A

The purpose of this appendix is to show how the results in [2] can be extended to apply to systems with more general mass matrices. That is, we relax the assumption that the matrix \mathbf{B}_0 is Eqn. (1.1) is given by $\mathbf{B}_0 = \mathbf{I}$ and replace it with the assumption that \mathbf{B}_0 is symmetric positive definite. More precisely, in this appendix, we show that if the results summarized in §2 hold for systems where \mathbf{B}_0 is the identity, matrix, then they hold for systems where \mathbf{B}_0 is a

symmetric positive definite matrix.

In order to apply the results with $\mathbf{B}_0 = \mathbf{I}$ to the case of more general \mathbf{B}_0 , we can introduce the vector

$$\mathbf{q} = \mathbf{B}_0^{1/2} \mathbf{x}. \quad (8.1)$$

The equation for \mathbf{q} can be written as

$$\frac{d\mathbf{q}}{dt} = \mathbf{B}_0^{-1/2} (\mathbf{A}_0 + \epsilon s(t) \mathbf{A}_1) \mathbf{B}_0^{-1/2} \mathbf{q}. \quad (8.2)$$

This is in the form of Eqn. (1.1). We need to know the eigenvalues of

$$\mathbf{C} = \mathbf{B}_0^{-1/2} \mathbf{A}_0 \mathbf{B}_0^{-1/2}. \quad (8.3)$$

It is a straightforward exercise to show that the eigenvectors of \mathbf{C} satisfy

$$\mathbf{q} = \mathbf{B}_0^{1/2} \mathbf{u} \quad \text{provided} \quad \mathbf{A}_0 \mathbf{u} = \lambda \mathbf{B}_0 \mathbf{u}. \quad (8.4)$$

Similarly, the adjoint eigenvectors of \mathbf{C} satisfy

$$\mathbf{p} = \mathbf{B}_0^{1/2} \mathbf{v} \quad \text{provided} \quad \mathbf{A}_0^T \mathbf{v} = \lambda \mathbf{B}_0 \mathbf{v}. \quad (8.5)$$

The eigenvectors \mathbf{p} and \mathbf{q} should be normalized so that

$$\overline{\mathbf{p}}_i^T \mathbf{q}_j = \delta_{ij}. \quad (8.6)$$

This is so provided

$$\overline{\mathbf{v}}_i^T \mathbf{B}_0 \mathbf{u}_j = \delta_{ij}. \quad (8.7)$$

That is, the eigenvectors \mathbf{u}_i and \mathbf{v}_j must be normalized using the inner product associated with \mathbf{B}_0 . In order to compute χ_{ij} , we need to compute

$$\chi_{ij} = \overline{\mathbf{v}}_i^T \mathbf{A}_1 \mathbf{u}_j. \quad (8.8)$$

This completes the proof that if the results in §2 hold for $\mathbf{B}_0 = \mathbf{I}$, then they hold for arbitrary symmetric positive definite \mathbf{B}_0 .

9 Appendix B

In this appendix, we discuss a particular choice of the function $G(z)$ that allows us to exercise the formulas in this paper. In the applications we are concerned with, the function $f(t)$ whose power spectral density we are evaluating is an acceleration. The power spectral density of the acceleration is ω^4 times the power spectral density of the displacement. For this reason, we require that our power spectral density is proportional to ω^4 for small values of ω .

Using the definition of $G(z)$ in Eqn. (1.2) and the fact that $R(\tau)$ is the inverse Fourier transform of the power spectral density $S(\omega)$, we see that

$$G(z) = \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty S(\omega) e^{i\omega\tau} e^{-\tau z} d\omega d\tau. \quad (9.1)$$

Reversing the order of integration and integrating with respect to τ gives

$$G(z) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{S(\omega)}{z - i\omega} d\omega. \quad (9.2)$$

We use the power spectral density

$$S(\omega) = \frac{\omega^4 a^3}{\pi A_{nor}} \left(\frac{1}{a^8 + (\omega - \omega_0)^8} + \frac{1}{a^8 + (\omega + \omega_0)^8} \right), \quad (9.3)$$

$$A_{nor} = \left((1 + \sqrt{2})\omega_0^4/a^4 + 6\omega_0^2/a^2 + 1 \right) \left(\sqrt{1 - 1/\sqrt{2}} \right). \quad (9.4)$$

This power spectral density has the property that $S(\omega) = S(-\omega)$, which corresponds to a real-valued autocorrelation function $R(\tau)$. It is normalized so that

$$\int_{-\infty}^{\infty} S(\omega) d\omega = 1. \quad (9.5)$$

For small values of a , this power spectral density produces narrow-band noise, with the energy concentrated around $\omega = \pm\omega_0$. For larger values of a , this power spectral density produces wide-band noise. With $S(\omega)$ chosen as in Eqn. (9.3), the function $G(z)$ can be evaluated using contour integration. We get

$$G(z) = \frac{1}{8i\pi a^4 A_{nor}} (G_0(z, \omega_0) + G_0(z, -\omega_0)), \quad (9.6)$$

where

$$G_0(z, \omega_0) = \sum_{k=0}^3 \frac{\gamma_k (a\gamma_k + \omega_0)^4}{z - i(a\gamma_k + \omega_0)} \quad \gamma_k = ie^{-3i\pi/8} e^{ik\pi/4}. \quad (9.7)$$

10 Appendix C

In this appendix, we show that the results in [2] apply to systems that have a KKT-like structure. That is, to systems of the form:

$$\mathbf{M}_0 \frac{d\mathbf{u}}{dt} = (\mathbf{K}_0 + \epsilon f(t)\mathbf{K}_1) \mathbf{u} + \mathbf{C}^T \mathbf{p}, \quad (10.1)$$

$$\mathbf{C}\mathbf{u} = 0. \quad (10.2)$$

Here, we assume that \mathbf{M}_0 is symmetric positive definite. Among other places, systems of this type appear when discretizing the equations of fluid dynamics. Using the matrices

$$\mathbf{B}_0 = \begin{pmatrix} \mathbf{M}_0 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0} \end{pmatrix}, \quad \mathbf{A}_0 = \begin{pmatrix} \mathbf{K}_0 & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{pmatrix}, \quad \mathbf{A}_1 = \begin{pmatrix} \mathbf{K}_1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad (10.3)$$

it is possible to write this system as in Eqn. (1.1), where the mass matrix \mathbf{B}_0 is singular. Since the resulting mass matrix is singular, it is not possible to directly apply the results in §2 and Appendix A. However, due to the special structure of this singular mass matrix, it is possible to extend the results of §2 and Appendix A so that they apply to the system of equations as in Eqns. (10.1) and (10.2). Thus, the purpose of this appendix is to show that the results in §2 apply to systems that have KKT-like structure as in Eqns. (10.3).

Assuming the matrix \mathbf{C} has fewer rows than columns, we can write

$$\mathbf{C}^T = \mathbf{Q}\mathbf{R}, \quad (10.4)$$

where \mathbf{R} is a non-singular matrix and \mathbf{Q} is a matrix whose columns are orthonormal. The projection matrix \mathbf{P} can be chosen to be a matrix whose columns are orthonormal and orthogonal to the columns of \mathbf{Q} . This gives us

$$\mathbf{Q}^T \mathbf{P} = 0, \quad (10.5)$$

$$\mathbf{P}^T \mathbf{P} = \mathbf{I} \quad (10.6)$$

It is clear that

$$\mathbf{P}^T \mathbf{q} = 0 \implies \exists \mathbf{p} \text{ such that } \mathbf{q} = \mathbf{C}^T \mathbf{p}, \quad (10.7)$$

$$\mathbf{C} \mathbf{u} = 0 \implies \exists \mathbf{z} \text{ such that } \mathbf{u} = \mathbf{P} \mathbf{z}. \quad (10.8)$$

Eqns. (10.7) and (10.8) show that we can write Eqns. (10.1) and (10.2) as

$$\mathbf{P}^T \left(\mathbf{M}_0 \mathbf{P} \frac{d\mathbf{z}}{dt} - (\mathbf{K}_0 + \epsilon f(t) \mathbf{K}_1) \mathbf{P} \mathbf{z} \right) = 0 \quad (10.9)$$

The matrix $\mathbf{P}^T \mathbf{M}_0 \mathbf{P}$ is non-singular; hence, we can apply the results for equations where \mathbf{M} is non-singular. We can now apply the results in §2 to Eqn. (10.9). If we do this, we get the eigenvalue problem

$$\sigma_k \mathbf{P}^T \mathbf{M}_0 \mathbf{P} = \mathbf{P}^T \mathbf{A}_0 \mathbf{P} \mathbf{z}_k. \quad (10.10)$$

If \mathbf{z}_k is an eigenvector of this eigenvalue problem, then we can define $\mathbf{u}_k = \mathbf{P} \mathbf{z}_k$, and \mathbf{p}_k such that

$$\sigma_k \mathbf{M}_0 \mathbf{u}_k = \mathbf{K}_0 \mathbf{u}_k + \mathbf{C}^T \mathbf{p}_k, \quad (10.11)$$

$$\mathbf{C} \mathbf{u}_k = 0, \quad (10.12)$$

$$\phi_k = \begin{pmatrix} \mathbf{P} \mathbf{z}_k \\ \mathbf{p}_k \end{pmatrix} = \begin{pmatrix} \mathbf{u}_k \\ \mathbf{p}_k \end{pmatrix}. \quad (10.13)$$

Thus, the vector ϕ_k is an eigenvector of the eigenvalue problem in Eqn. (2.1) with the matrices defined as in Eqn. (10.3). A similar result holds for the adjoint eigenvector ψ_k . It is straightforward to show that the eigenvectors and adjoint eigenvectors based on Eqn. (10.9) satisfy the same normalization conditions as in §2 and yield the same constant χ_{ij} provided we use the matrices defined as in Eqn. (10.3).

11 Appendix D

In this appendix, we give the dispersion relation for the eigenvalues associated with capillary gravity waves as discussed in §6. In particular, we consider Eqns. (6.2), (6.3), and (6.4), along with the boundary conditions in Eqns. (6.5), (6.6), (6.7), and (6.8). If we assume the temporal dependence of (u, w, p, h) is of the form $e^{\sigma t}$, we get an eigenvalue problem for σ . In this appendix, we give the one-dimensional transcendental equation for the eigenvalues σ of this system of equations.

In writing down the dispersion relation, we use the parameter

$$\tau = T\alpha^2 + \rho g_0. \quad (11.1)$$

We can write the dispersion relation for σ as $H(\sigma) = 0$, where

$$H(\sigma) = d_0 + \alpha d_s \sinh(\alpha L) \sin(\beta L) + \beta d_c \cosh(\alpha L) \cos(\beta L) + \tau d_\tau (\alpha \cosh(\alpha L) \sin(\beta L) - \beta \sinh(\alpha L) \cos(\beta L)) \quad (11.2)$$

and

$$d_0 = -4\alpha^2 \beta \nu (2\alpha^2 \nu + \sigma) \quad (11.3a)$$

$$d_c = \sigma^2 + 4\alpha^2 \nu \sigma + 8\alpha^4 \nu^2 \quad (11.3b)$$

$$d_s = -\sigma^2 - 8\alpha^2 \nu \sigma - 8\alpha^4 \nu^2 = -d_c - 4\alpha^2 \nu \sigma \quad (11.3c)$$

$$d_\tau = -\frac{\alpha}{\rho} \quad (11.3d)$$

and

$$\beta = i\sqrt{\frac{\sigma}{\nu} + \alpha^2}. \quad (11.4)$$

In order to use the results of Thm. 4.2, we write this as

$$H(\sigma) = H_0(\sigma) + \rho g_0 H_1(\sigma), \quad (11.5)$$

where

$$H_0(\sigma) = d_0 + \alpha d_s \sinh(\alpha L) \sin(\beta L) + \beta d_c \cosh(\alpha L) \cos(\beta L) + T\alpha^2 d_\tau (\alpha \cosh(\alpha L) \sin(\beta L) - \beta \sinh(\alpha L) \cos(\beta L)), \quad (11.6)$$

$$H_1(\sigma) = d_\tau (\alpha \cosh(\alpha L) \sin(\beta L) - \beta \sinh(\alpha L) \cos(\beta L)). \quad (11.7)$$

Dividing by $\rho H_1(\sigma)$, we get the dispersion relation

$$f(\sigma) = f_A(\sigma) + g_0, \quad (11.8)$$

where

$$f_A(\sigma) = \frac{H_0(\sigma)}{\rho H_1(\sigma)}. \quad (11.9)$$

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